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VERTEX DECOMPOSABILITY OF K-COHEN-MACAULAY SIMPLICIAL COMPLEXES OF CODIMENSION 3

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Abstract. Let Δ be a simplicial complex on vertex set $[n]$. It is shown that if Δ is k-Cohen–Macaulay of codimension 3, then Δ is vertex decomposable. As a consequence we show that Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$.

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1 Introduction

Let Δ be a simplicial complex on vertex set $[n] = \{1, \cdots, n\}$, i.e. Δ is a collection of subsets of $[n]$ with the the property that if $F \in \Delta$, then all subsets of F are also in Δ. An element of Δ is called a *face* of Δ, and the maximal faces of Δ under inclusion are called *facets*. We denote by $\mathcal{F}(\Delta)$ the set of facets of Δ. The *dimension* of a face F is defined as $\dim F = |F| - 1$, where $|F|$ is the number of vertices of F. The dimension of the simplicial complex Δ is the maximum dimension of its facets. A simplicial complex Δ is called *pure* if all facets of Δ have the same dimension. Otherwise it is called non-pure. We denote the simplicial complex Δ with facets $F_1, \dots, F_t$ by $\Delta = \langle F_1, \dots, F_t \rangle$. A simplex is a simplicial complex with only one facet.

For the simplicial complexes Δ_1 and Δ_2 defined on disjoint vertex sets, the join of Δ_1 and Δ_2 is $\Delta_1 * \Delta_2 = \{F \cup G \ : \ F \in \Delta_1,\ G \in \Delta_2\}$.

For the face F in Δ, the link, deletion and star of F in Δ are respectively, denoted by $\mathrm{link}_\Delta F$, $\Delta \setminus F$ and $\mathrm{star}_\Delta F$ and are defined by $\mathrm{link}_\Delta F = \{G \in \Delta \ : \ F \cap G = \varnothing,\ F \cup G \in \Delta\}$ and $\Delta \setminus F = \{G \in \Delta \ : \ F \nsubseteq G\}$ and $\mathrm{star}_\Delta F = \langle F \rangle * \mathrm{link}_\Delta F$.

Let $R = K[x_1, \dots, x_n]$ be the polynomial ring in n indeterminates over a field K. To a given simplicial complex Δ on the vertex set $[n]$, the Stanley–Reisner ideal is the squarefree monomial ideal whose generators correspond to the non-faces of Δ. we set:

$$\mathbf{x}_F = \prod_{x_i \in F} x_i.$$

We define the *facet ideal* of Δ, denoted by $I(\Delta)$, to be the ideal of S generated by $\{\mathbf{x}_F : \quad F \in \mathcal{F}(\Delta)\}$. The *non-face ideal* or the *Stanley-Reisner ideal* of Δ, denoted by I_Δ, is the ideal of S generated by square-free monomials $\{\mathbf{x}_F : \quad F \in \mathcal{N}(\Delta)\}$. Also we call $K[\Delta] := S/I_\Delta$ the *Stanley-Reisner ring* of Δ. We say the simplicial complex Δ is Cohen–Macaulay if $K[x_1, \dots, x_n]/I_\Delta$ is Cohen–Macaulay. One of interesting problems in combinatorial commutative algebra is the

Stanley's conjectures. The Stanley's conjectures are studied by many researchers. Let R be a $\mathbb{N}^n$- graded ring and M a $\mathbb{Z}^n$- graded R- module. Then Stanley (1982) conjectured that

$$\operatorname{depth}(M) \leq \operatorname{sdepth}(M)$$

He also conjectured in Stanley (1996) that each Cohen-Macaulay simplicial complex is partitionable. Herzog et al (2008) showed that the conjecture about partitionability is a special case of the Stanley's first conjecture. Duval et al (2016) construct a Cohen-Macaulay complex that is not partitionable, thus disproving the partitionability conjecture. Hachimori gived an open problem as following:

Whether every two dimensional Cohen-Macaulay simplicial complex is partitionable; see Hachimori, (2008). In this paper, we will answer to this problem in the special case. Ajdani & Bulnes (2019) proved the following result:

Theorem 1. *(Ajdani & Bulnes, 2019; Theorem 2.3) If Δ is a Cohen-Macaulay simplicial complex of codimension 2, then Δ is vertex decomposable.*

This paper is organized as follows: In Section 1, we recall some definitions and results which will be needed later. In Section 2, we show that any k-Cohen-Macaulay simplicial complex of codimension 3 is vertex decomposable (see Theorem 2).

In Section 3, as a consequence we show that Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$.

2 Preliminaries

In this section we recall some definitions and results which will be needed later.

Definition 1. *A simplicial complex Δ is recursively defined to be vertex decomposable, if it is either a simplex, or else has some vertex v so that,*

(a) *Both $\Delta \setminus v$ and $\operatorname{link}_\Delta(v)$ are vertex decomposable, and*

(b) *No face of $\operatorname{link}_\Delta(v)$ is a facet of $\Delta \setminus v$.*

A vertex v which satisfies in condition (b) is called a shedding vertex.

Definition 2. *A simplicial complex Δ is shellable, if the facets of Δ can be ordered $F_1, \ldots, F_s$ such that, for all $1 \leq i < j \leq s$, there exists some $v \in F_j \setminus F_i$ and some $l \in \{1, \ldots, j-1\}$ with $F_j \setminus F_l = \{v\}$.*

A simplicial complex Δ is called disconnected, if the vertex set V of Δ is a disjoint union $V = V_1 \cup V_2$ such that no face of Δ has vertices in both V_1 and V_2. Otherwise Δ is connected. It is well-known that

vertex decomposable $\Longrightarrow$ shellable $\Longrightarrow$ Cohen-Macaulay

Definition 3. *A graded S-module M is called sequentially Cohen-Macaulay (over K), if there exists a finite filtration of graded S-modules,*

$$0 = M_0 \subset M_1 \subset \cdots \subset M_r = M$$

such that each M_i/M_{i-1} is Cohen-Macaulay, and the Krull dimensions of the quotients are increasing:

$$\dim(M_1/M_0) < \dim(M_2/M_1) < \cdots < \dim(M_r/M_{r-1}).$$

3 Vertex decomposability k-Cohen-Macaulay simplicial complexes of codimension 3

As the main result of this section, it is shown that every k-Cohen-Macaulay simplicial complexes of codimension 3 is vertex decomposable. For the proof we need the following lemmas:

Lemma 1. *(Miyazaki, 1990; Lemma 2.3) Let Δ be a simplicial complex with vertex set V. Let $W \subseteq V$ and let σ be a face in Δ. If $W \cap \sigma = \varnothing$, then* $\mathrm{link}_{\Delta \setminus W}\{\sigma\} = \mathrm{link}_{\Delta}\{\sigma\} \setminus W$.

Definition 4. *Let K be a field. A simplicial complex Δ with vertex set V is called k-Cohen-Macaulay of dimension r over K if for any subset W of V (including $\varnothing$), $\Delta \setminus W$ is Cohen-Macaulay of dimension r over K.*

Lemma 2. *Let Δ be a simplicial complex with vertex set V. Then the following conditions are equivalent :*

(i) *Δ is k-Cohen-Macaulay;*

(ii) *for all $\sigma \in \Delta$,* $\mathrm{link}_{\Delta}\{\sigma\}$ *is k-Cohen-Macaulay;*

Proof. By lemma 1, for any subset W of V, we have $\mathrm{link}_{\Delta \setminus W}\{\sigma\} = \mathrm{link}_{\Delta}\{\sigma\} \setminus W$. Since $\Delta \setminus W$ is Cohen-Macaulay so $\mathrm{link}_{\Delta}\{\sigma\} \setminus W$ is Cohen-Macaulay. Therefore $\mathrm{link}_{\Delta}\{\sigma\}$ is k-Cohen-Macaulay. □

Now, we are ready that prove the main result of this section.

Theorem 2. *Let Δ be a k-Cohen-Macaulay simplicial complex of codimension 3 on vertex set $[n]$. Then Δ is vertex decomposable.*

Proof. We prove the theorem by induction on $|[n]|$ the number of vertices of Δ. If $|[n]| = 0$, then $\Delta = \{\}$ and it is vertex decomposable. Now Let $|[n]| > 0$ and $d \in [n]$ be a vertex of Δ. Then the simplicial complex $\mathrm{link}_{\Delta}\{d\}$ is a complex on $|[n]| - 1$ vertex and its dimension is $\dim \Delta - 1$. By lemma 2, $\mathrm{link}_{\Delta}\{d\}$ is again k-Cohen-Macaulay of codimension 3. Therefore by induction hypothesis $\mathrm{link}_{\Delta}\{d\}$ is vertex decomposable.

On the other hand since Δ is a k-Cohen-Macaulay, for each existing vertex $d \in \Delta$, $\Delta \setminus \{d\}$ is Cohen-Macaulay of codimension 2 and by Theorem 1, $\Delta \setminus \{d\}$ is vertex decomposable. It is easy to see that no face of $\mathrm{link}_{\Delta}\{d\}$ is a facet of $\Delta \setminus \{d\}$. Therefore any vertex d is a shedding vertex and Δ is vertex decomposable. □

4 Stanley Decompositions

Let R be any standard graded K- algebra over an infinite field K, *i.e*, R is a finitely generated graded algebra $R = \bigoplus_{i \geq 0} R_i$ such that $R_0 = K$ and R is generated by R_1. There are several characterizations of the depth of such an algebra. We use the one that depth(R) is the maximal length of a regular R- sequence consisting of linear forms. Let $x_F = \sqcap_{i \in F} x_i$ be a squarefree monomial for some $F \subseteq [n]$ and $Z \subseteq \{x_1, \ldots, x_n\}$. The K- subspace $x_F K[Z]$ of $S = K[x_1, \ldots, x_n]$ is the subspace generated by monomials $x_F u$, where u is a monomial in the polynomial ring $K[Z]$. It is called a squarefree Stanley space if $\{x_i : i \in F\} \subseteq Z$. The dimension of this Stanley space is $|Z|$. Let Δ be a simplicial complex on $\{x_1, \ldots, x_n\}$. A squarefree Stanley decomposition $\mathcal{D}$ of $K[\Delta]$ is a finite direct sum $\bigoplus_i u_i K[Z_i]$ of squarefree Stanley spaces which is isomorphic as a $\mathbb{Z}^n$- graded K- vector space to $K[\Delta]$, *i.e.*

$$K[\Delta] \cong \bigoplus_i u_i K[Z_i].$$

We denote by sdepth($\mathcal{D}$) the minimal dimension of a Stanley space in $\mathcal{D}$ and we define sdepth($K[\Delta]$) = max{sdepth($\mathcal{D}$)}, where $\mathcal{D}$ is a Stanley decomposition of $K[\Delta]$. Stanley conjectured in Stanley (1982) the upper bound for the depth of $K[\Delta]$ as the following:

$$\mathrm{depth}(K[\Delta]) \leq \mathrm{sdepth}(K[\Delta]).$$

Also we recall another conjecture of Stanley. Let Δ be again a simplicial complex on $\{x_1, \ldots, x_n\}$ with facets $G_1, \ldots, G_t$. The complex Δ is called partitionable if there exists a partition $\Delta = \bigcup_{i=1}^{t}[F_i, G_i]$ where $F_i \subseteq G_i$ are suitable faces of Δ. Here the interval $[F_i, G_i]$ is the set of faces $\{H \in \Delta : F_i \subseteq H \subseteq G_i\}$. In Stanley (1996) and Stanley (2000) respectively Stanley conjectured each Cohen-Macaulay simplicial complex is partitionable. This conjecture is a special case of the previous conjecture. Indeed, Herzog et al (2008) proved that for Cohen-Macaulay simplicial complex Δ on $\{x_1, \ldots, x_n\}$ we have that depth($K[\Delta]$) $\leq$ sdepth($K[\Delta]$) if and only if Δ is partitionable. Since each vertex decomposable simplicial complex is shellable and each shellable complex is partitionable. Then as a consequence of our results we obtain :

Corollary 1. *If Δ is a k-Cohen-Macaulay simplicial complex of codimension 3, then Δ is partitionable and Stanley's conjecture holds for $K[\Delta]$.*

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